

PREPARED FOR SUBMISSION TO JHEP

Nonextensive Friedmann equations and new bounds for Tsallis parameter through noncommutative entropic gravity

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March 2, 2013

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ABSTRACT: In this paper, we analyze the nonextensive Tsallis' statistical mechanics in the light of Verlinde's formalism. As an important result we obtain, with the aid of a noncommutative entropic gravity, a new bound for Tsallis nonextensive parameter that is clearly different from the one present in the current literature. We also derive the Friedmann equations in a nonextensive scenario.

KEYWORDS: Models of Quantum Gravity, Cosmology of Theories beyond the SM

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1 Introduction

In principle, gravitation has been greatly benefited from a possible connection with thermodynamics. Pioneering works of Bekenstein [1] and Hawking [2] have described the issue. For example, quantities as area and mass of black-holes are associated with entropy and temperature respectively. Keeping in the same line, Jacobson [3] interpreted Einstein field equations as a thermodynamic identity. Padmanabhan [4] gave an interpretation of gravity as an equipartition theorem.

Recently, Verlinde [5] gave a heuristic derivation of gravity, both Newtonian and relativistic, at least for static spacetime, being the equipartition law of energy also playing an important role. On the other hand, one can ask what is the contribution of gravitational models in thermodynamics theories? There is an extension of the usual Boltzmann-Gibbs theory (BG) that is called Tsallis' thermostatistics [6], (TT). This formalism initially considers the entropy formula as a nonextensive quantity where there is a parameter q that measure the degree of nonextensivity. This formalism has been successfully applied in many physical models. When $q \rightarrow 1$ we return to the usual Boltzmann-Gibbs theory, i.e., we have a extensive theory.

The purpose of this paper is to use the Verlinde's formalism in a noncommutative scenario [7] in order to determine a new bound for the nonextensive parameter q . We will see that when we make the equality between the gravitational constant modified by

noncommutative effects and the gravitational constant modified by nonextensive effects, a new and more precise bound for the nonextensive parameter q appears. We shall structure our paper in the following way: In Section 2 we give a brief review of Verlinde's formalism. In Section 3 we introduce main steps of the Tsallis' approach of thermostatics. In Section 4 we derive the new bound for the nonextensive parameter q . In the Section 5 we derive the Friedmann equation in a nonextensive scenario. The conclusions depicted in the last section.

2 A Brief Review of Verlinde's Formalism

The formalism proposed by E. Verlinde [5] derives the gravitational acceleration by using, basically, the holographic principle and the equipartition law of energy. This model considers a spherical surface as the holographic screen, with a particle of mass M positioned in its center. A holographic screen can be imagined as a storage device for information. The number of bits (the term bit means the smallest unit of information in the holographic screen) is assumed to be proportional to the area A of the holographic screen

$$N = \frac{A}{l_p^2}, \quad (2.1)$$

where $A = 4\pi r^2$ and $l_p = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length. In Verlinde's formalism we assume that the total energy of the bits on the screen is given by the equipartition law of energy

$$E = \frac{1}{2} N k_B T. \quad (2.2)$$

It is important to mention here that the usual equipartition theorem, Eq.(2.2), is derived from the usual Boltzmann-Gibbs thermostatics. In a nonextensive thermostatics scenario, the equipartition law of energy will be modified in a sense that a nonextensive parameter q will be introduced in its expression. Considering that the energy of the particle inside the holographic screen is equally divided through all bits then we can write the equation

$$Mc^2 = \frac{1}{2} N k_B T. \quad (2.3)$$

Using Eq. (2.1), and the Unruh temperature formula [8]

$$k_B T = \frac{1}{2\pi} \frac{\hbar a}{c}, \quad (2.4)$$

we are in a position to derive the (absolute) gravitational acceleration formula

$$\begin{aligned} a &= \frac{l_p^2 c^3}{\hbar} \frac{M}{r^2} \\ &= G \frac{M}{r^2}. \end{aligned} \quad (2.5)$$

We can observe that from Eq. (2.5) the Newton constant G is just written in terms of the fundamental constants, $G = \frac{l_p^2 c^3}{\hbar}$.

3 The Tsallis' Thermostatistics

An important formulation of a nonextensive (NE) Boltzmann-Gibbs thermostatistics has been proposed by Tsallis [6] in which the entropy is given by the formula

$$S_q = k_B \frac{1 - \sum_{i=1}^W p_i^q}{q - 1} \quad \left(\sum_{i=1}^W p_i = 1 \right), \quad (3.1)$$

where p_i is the probability of the system to be in a microstate, W is the total number of configurations and q , known in the current literature as Tsallis parameter or nonextensive parameter, is a real parameter quantifying the degree of nonextensivity. The definition of entropy (3.1) has as motivation to study multifractal systems and it also possesses the usual properties of positivity, equiprobability, concavity and irreversibility. It is important to note that Tsallis' formalism contains the Boltzmann-Gibbs statistics as a particular case in the limit $q \rightarrow 1$ where the usual additivity of entropy is recovered. Plastino and Lima [9] by using a generalized velocity distribution for free particles [10]

$$f_0(v) = B_q \left[1 - (1 - q) \frac{mv^2}{2k_B T} \right]^{1/(1-q)}, \quad (3.2)$$

where B_q is a q -dependent normalization constant, m and v is a mass and velocity of the particle, respectively. They have derived a nonextensive equipartition law of energy whose expression is given by

$$E = \frac{1}{5 - 3q} N k_B T, \quad (3.3)$$

where the range of q is $0 \leq q < 5/3$. For $q = 5/3$ (critical value) the expression of the equipartition law of energy, Eq. (3.3), diverges. It is easy to observe that for $q = 1$, the classical equipartition theorem for each microscopic degrees of freedom is recovered. It is important to mention that the virial theorem is not modified in this nonextensive thermostatistics formalism [11].

3.1 The Nonextensive Equipartition Theorem and Its Application in the Verlinde's Formalism

Substituting the equipartition law by the nonextensive equipartition formula, Eq. (3.3), in the equation (2.3), and applying the same steps described in section 2, we can obtain a modified acceleration formula given by

$$a = G_{NE} \frac{M}{r^2}, \quad (3.1)$$

where G_{NE} is an effective gravitational constant which is written as

$$G_{NE} = \frac{5 - 3q}{2} G. \quad (3.2)$$

From result (3.2) we can observe that the effective gravitational constant depends on the nonextensive parameter q . For example, $q = 1$ we have $G_{NE} = G$ and for $q = \frac{5}{3}$ we have a curious and hypothetical value that is $G_{NE} = 0$.

4 A New precise Bound for the Nonextensivity Parameter through Non-commutativity

It is nowadays very well establish that to fathom the noncommutative (NC) quantum theory can lead us to understand the details of early Universe physics [12]. Noncommutativity was rekindled when it was also realized in superstring/M-theory. A NC algebra arises when describing the excitations of open strings in the presence of a Neveu-Schwarz constant background field [13]. We can also see the noncommutativity in other areas of research such as entropic gravity [7, 14], quantum cosmology [15] the Schwarzschild black hole thermodynamics [16] and the black hole singularity [17].

In this section we make use of the results obtained in [7] to obtain a new bound for the nonextensive parameter using NC space, which is a new result. First of all we will consider that the geometry that governs our space is based on a canonical phase-space NC algebra. Let us describe our d -dimensional phase-space such as [7],

$$[\hat{q}_i, \hat{q}_j] = i\theta_{ij} \quad , \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad , \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij} \quad i, j = 1, \dots, d \quad (4.1)$$

where η_{ij} and θ_{ij} are antisymmetric real constant $d \times d$ matrices and δ_{ij} is the identity matrix.

Now we will describe the NC correction to the gravitational force developed in [7]. The basic idea is that the Planck constant is corrected by NC effects. We begin by considering that a two dimensional phase-space cell has a minimal volume \hbar^2 given by

$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) = \Delta x_1 \Delta p_1 \Delta x_2 \Delta p_2 \quad (4.2)$$

where $\Delta x_1 \Delta p_1 \geq \frac{\hbar}{2}$, $\Delta x_2 \Delta p_2 \geq \frac{\hbar}{2}$, $\Delta x_1 \Delta x_2 \geq \frac{\theta_{ij}}{2}$ and $\Delta p_1 \Delta p_2 \geq \frac{\eta_{ij}}{2}$ are the constraints. For more details see [7]. If we want to minimize the product $\Delta x_1 \Delta x_2 = \frac{\theta_{ij}}{2}$ we have to use that $\Delta p_1 \Delta p_2 = \frac{\eta_{ij}}{2}$, hence,

$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) \geq \frac{\theta_{12}\eta_{12}}{4} \quad (4.3)$$

To minimize $\Delta x_1 \Delta p_1$ we have to use that $\Delta x_1 \Delta p_1 = \frac{\hbar}{2} = \Delta x_2 \Delta p_2$, so we can write that,

$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) \geq \frac{\hbar^2}{4} \quad (4.4)$$

and the total volume is

$$V(\Delta x_1, \Delta p_1, \Delta x_2, \Delta p_2) \geq \frac{\hbar^2}{4} + \frac{\theta_{12}\eta_{12}}{4} = \frac{\hbar_{eff}^2}{4} \quad (4.5)$$

$$\Rightarrow \quad \hbar_{eff} = \hbar \sqrt{1 + \frac{\theta_{12}\eta_{12}}{\hbar^2}} \quad (4.6)$$

$$\Rightarrow \quad F_{NC} = F \hbar \sqrt{1 + \frac{\theta_{12}\eta_{12}}{\hbar^2}} = \frac{GMm}{r^2} \sqrt{1 + \frac{\theta_{12}\eta_{12}}{\hbar^2}} \quad (4.7)$$

which is the NC corrections for the inverse square law.

The cherished point in our work concerns the bounds on the Tsallis nonextensive parameter q . In order to derive this, we notice that an effective G derived by NC effects is written as

$$G_{NC} = G \frac{\hbar_{eff}}{\hbar}. \quad (4.8)$$

Equating the effective G_{NC} coming from the NC correction, Eq.(4.8), with G_{NE} from the NE correction, Eq.(3.2), we have

$$G \frac{\hbar_{eff}}{\hbar} = \frac{G}{2} (5 - 3q). \quad (4.9)$$

So, we obtain

$$\hbar_{eff} = \frac{\hbar}{2} (5 - 3q). \quad (4.10)$$

Expanding Eq. (4.6) we can write that

$$\hbar_{eff} \approx \hbar + \frac{\theta\eta}{2\hbar} = \hbar \left(1 + \frac{\theta\eta}{2\hbar^2}\right). \quad (4.11)$$

Substituting this result in (4.10), we obtain

$$\begin{aligned} 1 + \frac{\theta\eta}{2\hbar^2} &= \frac{1}{2} (5 - 3q) \\ \Rightarrow \frac{\theta\eta}{3\hbar^2} &= 1 - q. \end{aligned} \quad (4.12)$$

Due to the experimental results in [19] we can write that [7]

$$\frac{\theta\eta}{\hbar} \leq O(1) \cdot 10^{-13}. \quad (4.13)$$

So

$$|q - 1| \leq O(1) \cdot 10^{-13}, \quad (4.14)$$

which establish a new bound for the nonextensive parameter. This value is 10^9 times below the bounds obtained in [20] and [21].

At this point we would like to mention that when we impose the equality (4.9), we have assumed that both NC and NE corrections to the gravitational constant are the same. This assumption is based on the fact that NC and NE effects have a common origin that is the spacetime, in a short distance (order of Planck length), displaying a possible fractal structure. Therefore, we are considering that the usual Boltzmann-Gibbs thermodynamics with NC variables are equivalent to nonextensive thermodynamics with commutative variables. A more detailed discussion about this correspondence can be found, for example, in the references [22] and [23].

5 Friedmann Equations Using Tsallis' Statistics

In order to make this paper self-contained, in the first part of this section we review the main steps of the obtainment of the Friedmann equations through the entropic force basics. This calculation is followed by the obtainment of the same equation through the nonextensive Tsallis statistics. Our objective is to obtain the same expression for the gravitational constant obtained in the last section. We think that this result confirms the relation between both constants.

In the following we will review the calculation performed in [24] where the Friedmann equations, which rules the dynamical evolution of the Friedmann-Robertson-Walker Universe, were obtained through the entropic force basics together with the equipartition law of energy. The authors also used the concept of Unruh temperature combined with Verlinde's work.

5.1 Friedmann Equations from Entropic Force: a quick review

Let us consider the FRW metric given by

$$ds^2 = dt^2 + a^2(t) (dr^2 + r^2 d\Omega^2), \quad (5.1)$$

where $a(t)$ is the scale factor of the Universe. Following Verlinde's point of view [5], we will consider a compact spatial region Λ with a compact boundary $\partial\Lambda$, which is a sphere with physical radius $\tilde{r} = ar$. In this framework, this compact boundary plays the role of the holographic screen. Relativistically we can write that $E = Mc^2$, where M represents the mass that would emerge in the compact spatial region Λ surrounded by the boundary screen $\partial\Lambda$.

Assuming that the FRW Universe is a perfect fluid with stress-energy tensor given by

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \quad (5.2)$$

and the total mass is not conserved anymore. We can consider that the change in the total mass is equal to the work given by the pressure so that $dM = -pdV$. Consequently we have the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \text{where} \quad H = \frac{\dot{a}}{a} \quad (\text{Hubble parameter}). \quad (5.3)$$

The total mass inside the spatial region Λ is given by

$$M = \int_{\Lambda} dV T_{\mu\nu} u^{\mu} u^{\nu}, \quad (5.4)$$

where $T_{\mu\nu} u^{\mu} u^{\nu}$ is energy density measured by a comoving observer. A comoving observer at the place of the screen r can measure the acceleration as

$$a_r = -\frac{d^2 \tilde{r}}{dt^2} = -\ddot{a} r. \quad (5.5)$$

This acceleration is caused by the matter in the spatial region enclosed by the boundary $\partial\Lambda$

The well known Unruh formula is given by

$$T = \frac{1}{2\pi k_B c} \hbar a_r \Rightarrow k_B T = -\frac{1}{2\pi c} \hbar \ddot{a} r \quad (5.6)$$

where we can see the relation between the acceleration and the temperature. Substituting (2.1), (5.6) and the relativistic energy in (2.2) we obtain that

$$\ddot{a} = -\frac{4\pi G}{3} \rho a, \quad (5.7)$$

which is the holographic way to derive the dynamical equation for Newtonian cosmology [24]. Let us define the active gravitational mass \mathcal{M} , different from the total mass in the spatial region Λ . It is the well known Tolman-Komar mass, defined as

$$\mathcal{M} = 2 \int_{\Lambda} dV \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) u^{\mu} u^{\nu}, \quad (5.8)$$

which can be rewritten as

$$\mathcal{M} = 2 \left[(\rho + p) - \frac{\hbar r}{4\pi k_B c} \ddot{a} \right] \frac{4\pi \tilde{r}^3}{3}, \quad (5.9)$$

and finally, replacing M by \mathcal{M} we can write in this case that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p). \quad (5.10)$$

This is the defined acceleration equation for the dynamical evolution of the FRW Universe. Using the continuity equation and after an integration we can obtain the Friedmann equations for the FRW Universe given by

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \quad (5.11)$$

where k is an integration constant which can be realized as the spatial, curvature in the region Λ in the Einstein theory of general relativity. The possibilities for k defines the geometry of the FRW Universe and they are the well known 1 (closed), 0 (flat) and -1 (open).

The reader can see more details in [5] about the extension of the above calculation for the case of extra dimensions, i.e., $d \geq 4$. For examples, the number of bits on the screen now is

$$N = \frac{d-2}{d-3} \frac{Ac^3}{G\hbar}, \quad (5.12)$$

and the continuity equation is

$$\dot{\rho} + (d-1)H(\rho + p) = 0. \quad (5.13)$$

The active mass is

$$\mathcal{M} = \frac{d-2}{d-3} \int_{\Lambda} dV \left(T_{\mu\nu} - \frac{1}{d-2} T g_{\mu\nu} \right) u^{\mu} u^{\nu}, \quad (5.14)$$

and the acceleration given in (5.10) is now written as

$$\frac{\ddot{a}}{a} = - \frac{8\pi G}{(d-1)(d-2)} [(d-3)\rho + (d-1)p]. \quad (5.15)$$

Finally, the Friedmann equation of the FRW Universe can be represented in d dimensions by

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{(d-1)(d-2)}. \quad (5.16)$$

This is the holographic principle derivation of the Friedmann equations of the FRW Universe. We used strictly the entropic ideas derived by Verlinde through the equipartition law of energy. The main argument obviously is that gravity appears as an entropic force.

In the following, as explained before, we will obtain these above equations using the nonextensive ideas developed by Tsallis.

5.2 The Modified Friedmann Equations through Tsallis Principles

For the nonextensive (NE) concept of equipartition law of energy we have that

$$E_{NE} = \frac{1}{5-3q} NkT. \quad (5.17)$$

Following the same calculation steps developed in the last Subsection with the nonextensive equipartition formula given by (5.17), we can obtain the nonextensive acceleration equation for the dynamical evolution of the FRW Universe

$$\frac{\ddot{a}}{a} = -(5-3q) \frac{2\pi G}{3} (\rho + 3p), \quad (5.18)$$

and in a direct calculation we can write the (nonextensive) Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{4(5-3q)\pi G}{3} \rho a^2, \quad (5.19)$$

which is easy to make a comparison with Eq. (3.2) and see that

$$G_{NE} = G \frac{(5-3q)}{2}, \quad (5.20)$$

which is the same result obtained in the subsection 3.1. So, we see this result as a confirmation that the nonextensive equipartition law of energy introduces the nonextensive parameter in the gravitational constant. We can understand Eq. (5.20) as a numerical map between both constants in such a way to produce a direct conversion from one concept to another. In other words we can say that a simple substitution determined by Eq. (5.20) produces introduction of Tsallis' concept in any theory. For example, it is easy to obtain the dimensional generalization of the Friedmann equation in d dimension. This equation can be written as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{NE}}{(d-1)(d-2)} \rho, \quad (5.21)$$

and G_{NE} is given by Eq. (5.20).

6 Conclusions

We have applied the nonextensive equipartition law of energy in Verlinde's formalism of gravity. As a result we have obtained an effective gravitational constant that incorporates effects of the nonextensive thermostatics developed by Tsallis. On the other side, the noncommutative gravity theory also predicts an effective gravitational constant.

If we consider both gravitational constants equal, a new bound for the Tsallis q -parameter can be derived. The result of the protocol used to obtain a nonextensive Friedmann equations is simply to write the usual gravitational constant as a function of the nonextensive effective gravitational constant.

7 Acknowledgments

WO would like to thank CNPq, Conselho Nacional de Desenvolvimento Científico e Tecnológico, brazilian research support agency, for financial support.

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